CLASSIFICATION OF SIMPLE QUARTICS UP TO EQUISINGULAR DEFORMATION

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Abstract. We study complex spatial quartic surfaces with simple singularities up to equisingular deformations; as a first step, give a complete equisingular deformation classification of the so-called non-special simple quartic surfaces.

1. Introduction

1.1. **Motivation.** Throughout the paper, all algebraic varieties are over the field $\mathbb{C}$ of complex numbers. A *quartic* is a surface in $\mathbb{P}^3$ of degree 4. We confine ourselves to *simple* quartics only, *i.e.*, those with **A**–**D**–**E** type singularities (see §3.1). Two such quartics are said to be *equisingular deformation equivalent* if they belong to the same deformation family in which the total Milnor number stays constant.

The theory of simple spatial quartics is very similar to theory of simple plane sextics: both are closely related to $K3$-surfaces. It is well known that the minimal resolution of singularities of a simple quartic $X \subset \mathbb{P}^3$ is a $K3$-surface. By using the global Torelli theorem for $K3$-surfaces [25] and the surjectivity of the period map [16] the equisingular deformation classification of simple quartics can be reduced to the study of a certain arithmetic question about lattices. The corresponding counterpart for plane sextics is covered by A. Akyol and A. Degtyarev [1] (see also [5]), who completed the equisingular deformation classification of irreducible singular plane sextics.

Found in the literature are a great number of papers dealing with quartics in $\mathbb{P}^3$ and based on the theory of $K3$-surfaces and V. V. Nikulin's results [22] on lattice extensions. For example, T. Urabe [29, 30] listed (in terms of perturbations of Dynkin graphs) all sets of singularities with the total Milnor number $\mu \leq 17$ that are realized by simple quartics. He also showed in [29] that $\mu \leq 19$ for a simple quartic. Note that the classification of non-simple quartics with isolated singularities (which is not related to $K3$-surfaces) is complete: a complete list and a description in terms of lattice embeddings are found in A. Degtyarev [3, 4], and a description of some realizable sets of singularities in terms of Dynkin diagrams is found in T. Urabe [27, 28].

Also worth mentioning are various $K3$-related deformation classification problems dealing with real surfaces and other polarizations; see, e.g., [8, 14, 23] and the survey [9] for further references. Specifically, real quartics in $\mathbb{P}^3$ have been addressed by V. Kharlamov [15] (the classification of nonsingular real quartics up to rigid isotopy) and A. Degtyarev, I. Itenberg [7] (arrangements of the ten nodes of a generic real determinantal quartic).

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1.2. **Principal results.** In this paper we confine ourselves to the so-called *non-special* quartics (the precise definition is too technical to be stated here and we refer §3.3). The counterpart of this notion in the realm of plane sextics are irreducible sextics admitting no dihedral coverings, *cf.* [1]. As yet another motivation, we have the following geometric characterization.

1.2.1. **Theorem.** *A simple quartic $X \subset \mathbb{P}^3$ is non-special if and only if*

$$H_1(X \smallsetminus (\operatorname{Sing} X \cup H)) = 0,$$

where $\operatorname{Sing} X$ *is the set of the singular points of X and H is a generic hyperplane section of X.*

This theorem is proved in §4.1.

A set of simple singularities can be identified with a root system, *i.e.*, a negative definite lattice generated by vectors of square -2 (see Dufree [10] and §3.1). By a *perturbation* of a set of simple singularities $\mathbf{S}$ we mean any set of simple singularities $\mathbf{S}'$ whose Dynkin graph is an induced subgraph of that of $\mathbf{S}$ (see §4.2). Recall that for a simple quartic $X \subset \mathbb{P}^3$, one has $\mu(X) \leq 19$ (see *e.g.*, [29]); X is called *maximizing* if $\mu(X) = 19$.

Denote by $\mathcal{M}(\mathbf{S})$ the equisingular stratum of simple quartics with a given set of singularities $\mathbf{S}$. A connected component $\mathcal{D} \subset \mathcal{M}(\mathbf{S})$ is called *real* if it is preserved as a set under the complex conjugation map $\operatorname{conj} : \mathbb{P}^3 \to \mathbb{P}^3$. Clearly, this property is independent of the choice of coordinates in $\mathbb{P}^3$, and all components of $\mathcal{M}(\mathbf{S})$ split into real and pairs of complex conjugate ones.

Our principal result is a complete description of the equisingular strata $\mathcal{M}_1(\mathbf{S})$ of non-special simple quartics.

1.2.2. **Theorem.** *A set of singularities $\mathbf{S}$ is realizable as the set of singularities of a non-special simple quartic if and only if $\mathbf{S}$ can be obtained by a perturbation from one of those listed in Tables* 1 *and* 2*. The numbers (r, c) of, respectively, real and pairs of complex conjugate components of the strata $\mathcal{M}_1(\mathbf{S})$ with $\mu(\mathbf{S}) = 19$ are shown in Table* 1*. If $\mathbf{S}$ is one of*

$$\mathbf{D}_6 \oplus 2\mathbf{A}_6, \quad \mathbf{D}_5 \oplus 2\mathbf{A}_6 \oplus \mathbf{A}_1, \quad 2\mathbf{A}_7 \oplus 2\mathbf{A}_2, \quad 3\mathbf{A}_6, \quad 2\mathbf{A}_6 \oplus 2\mathbf{A}_3$$

then $\mathcal{M}_1(\mathbf{S})$ consists of two complex conjugate components; in all other cases, the stratum $\mathcal{M}_1(\mathbf{S})$ is connected.

Theorem 1.2.2 is proved in §4.2.

1.3. **Contents of the paper.** Our principal result, Theorem 1.2.2, is proved by a reduction to an arithmetical problem [7] (*cf.* also [5]), followed by Nikulin's theory of lattice extensions via discriminant groups [22], Nikulin's existence theorem [22], and Miranda–Morrison theory [18, 19, 20] computing the genus groups and a few other bits missing in [22] in the case of indefinite lattices.

In §2, based on Nikulin's work [22], we recall the basic notions and results about integral lattices, discriminant forms and lattice extensions; then, we outline the fundamentals of Miranda-Morison's theory [20] which are used in §4.2. In §3, we discuss the relation between simple quartics and $K3$-surfaces, explain the notion of abstract homological type, and recall the reduction of the classification problem to the arithmetical classification of abstract homological types. Finally, §4 is devoted to the proofs of our principal results: the proof of Theorem1.2.2 is purely homotopy

Table 1. The space $\mathcal{M}_1(\mathbf{S})$ with $\mu(\mathbf{S}) = 19$

Singularities	(r, c)	Singularities	(r, c)
$2\mathbf{E}_8 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1, 0)$	$\mathbf{A}_{18} \oplus \mathbf{A}_1$	$(1, 1)$
$\mathbf{E}_8 \oplus \mathbf{E}_7 \oplus \mathbf{A}_4$	$(1, 0)$	$\mathbf{A}_{17} \oplus \mathbf{A}_2$	$(1, 1)$
$\mathbf{E}_8 \oplus \mathbf{E}_6 \oplus \mathbf{D}_5$	$(1, 0)$	$\mathbf{A}_{16} \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1, 0)$
$\mathbf{E}_8 \oplus \mathbf{E}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	$(1, 0)$	$\mathbf{A}_{15} \oplus 2\mathbf{A}_2$	$(0, 1)$
$\mathbf{E}_8 \oplus \mathbf{D}_7 \oplus 2\mathbf{A}_2$	$(1, 0)$	$\mathbf{A}_{14} \oplus \mathbf{A}_5$	$(0, 2)$
$\mathbf{E}_8 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_1$	$(1, 0)$	$\mathbf{A}_{14} \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$	$(0, 2)$
$\mathbf{E}_8 \oplus \mathbf{A}_9 \oplus \mathbf{A}_2$	$(1, 0)$	$\mathbf{A}_{13} \oplus \mathbf{A}_6$	$(0, 2)$
$\mathbf{E}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_5$	$(1, 0)$	$\mathbf{A}_{13} \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(1, 0)$
$\mathbf{E}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	$(1, 0)$	$\mathbf{A}_{12} \oplus \mathbf{A}_6 \oplus \mathbf{A}_1$	$(1, 1)$
$\mathbf{E}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$	$(0, 1)$	$\mathbf{A}_{12} \oplus \mathbf{A}_5 \oplus \mathbf{A}_2$	$(1, 1)$
$\mathbf{E}_8 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1, 0)$	$\mathbf{A}_{12} \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(0, 1)$
$\mathbf{E}_7 \oplus \mathbf{E}_6 \oplus \mathbf{A}_6$	$(1, 0)$	$\mathbf{A}_{12} \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$	$(2, 0)$
$\mathbf{E}_7 \oplus \mathbf{A}_{12}$	$(1, 1)$	$\mathbf{A}_{11} \oplus \mathbf{A}_6 \oplus \mathbf{A}_2$	$(0, 2)$
$\mathbf{E}_7 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_2$	$(0, 1)$	$\mathbf{A}_{10} \oplus \mathbf{A}_9$	$(1, 1)$
$\mathbf{E}_7 \oplus \mathbf{A}_8 \oplus \mathbf{A}_4$	$(2, 0)$	$\mathbf{A}_{10} \oplus \mathbf{A}_8 \oplus \mathbf{A}_1$	$(0, 1)$
$\mathbf{E}_7 \oplus 2\mathbf{A}_6$	$(0, 1)$	$\mathbf{A}_{10} \oplus \mathbf{A}_7 \oplus \mathbf{A}_2$	$(0, 2)$
$\mathbf{E}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(1, 0)$	$\mathbf{A}_{10} \oplus \mathbf{A}_6 \oplus \mathbf{A}_3$	$(0, 2)$
$2\mathbf{E}_6 \oplus \mathbf{D}_7$	$(1, 0)$	$\mathbf{A}_{10} \oplus \mathbf{A}_6 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(1, 0)$
$\mathbf{E}_6 \oplus \mathbf{D}_{13}$	$(1, 0)$	$\mathbf{A}_{10} \oplus \mathbf{A}_5 \oplus \mathbf{A}_4$	$(1, 0)$
$\mathbf{E}_6 \oplus \mathbf{D}_9 \oplus \mathbf{A}_4$	$(1, 0)$	$\mathbf{A}_{10} \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$	$(0, 1)$
$\mathbf{E}_6 \oplus \mathbf{A}_{13}$	$(1, 0)$	$\mathbf{A}_9 \oplus \mathbf{A}_8 \oplus \mathbf{A}_2$	$(1, 1)$
$\mathbf{E}_6 \oplus \mathbf{A}_{12} \oplus \mathbf{A}_1$	$(1, 0)$	$\mathbf{A}_9 \oplus \mathbf{A}_6 \oplus 2\mathbf{A}_2$	$(1, 0)$
$\mathbf{D}_{15} \oplus 2\mathbf{A}_2$	$(1, 0)$	$\mathbf{A}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_5$	$(1, 1)$
$\mathbf{D}_{11} \oplus \mathbf{A}_6 \oplus \mathbf{A}_2$	$(0, 1)$	$\mathbf{A}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_1$	$(0, 1)$
$\mathbf{D}_9 \oplus \mathbf{A}_6 \oplus 2\mathbf{A}_2$	$(1, 0)$	$\mathbf{A}_8 \oplus \mathbf{A}_6 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$	$(0, 3)$
$\mathbf{D}_7 \oplus \mathbf{A}_{10} \oplus \mathbf{A}_2$	$(0, 1)$	$\mathbf{A}_7 \oplus 2\mathbf{A}_6$	$(0, 2)$
$\mathbf{D}_7 \oplus 2\mathbf{A}_6$	$(0, 1)$	$\mathbf{A}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(0, 1)$
$\mathbf{D}_7 \oplus \mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2$	$(0, 1)$	$2\mathbf{A}_6 \oplus \mathbf{A}_5 \oplus \mathbf{A}_2$	$(2, 0)$
$\mathbf{D}_7 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2$	$(1, 0)$	$2\mathbf{A}_6 \oplus \mathbf{A}_4 \oplus \mathbf{A}_2 \oplus \mathbf{A}_1$	$(0, 1)$
		$\mathbf{A}_6 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_3 \oplus \mathbf{A}_2$	$(2, 0)$

Table 2. Extremal sets of singularities with $\mu(\mathbf{S}) = 18$

$\mathbf{E}_8 \oplus \mathbf{D}_{10}$	$\mathbf{D}_{14} \oplus \mathbf{A}_4$	$2\mathbf{D}_5 \oplus \mathbf{A}_8$
$\mathbf{E}_8 \oplus \mathbf{D}_9 \oplus \mathbf{A}_1$	$\mathbf{D}_{10} \oplus \mathbf{A}_8$	$2\mathbf{D}_5 \oplus 2\mathbf{A}_4$
$2\mathbf{E}_7 \oplus 2\mathbf{A}_2$	$\mathbf{D}_{10} \oplus 2\mathbf{A}_4$	$\mathbf{D}_5 \oplus \mathbf{A}_9 \oplus \mathbf{A}_4$
$\mathbf{E}_7 \oplus \mathbf{D}_{11}$	$2\mathbf{D}_9$	$\mathbf{D}_5 \oplus \mathbf{A}_8 \oplus \mathbf{A}_5$
$\mathbf{E}_7 \oplus \mathbf{D}_9 \oplus \mathbf{A}_2$	$\mathbf{D}_9 \oplus \mathbf{A}_8 \oplus \mathbf{A}_1$	$\mathbf{D}_5 \oplus \mathbf{A}_5 \oplus 2\mathbf{A}_4$
$\mathbf{D}_{18}$	$\mathbf{D}_6 \oplus 3\mathbf{A}_4$	$2\mathbf{A}_9$
$\mathbf{D}_{17} \oplus \mathbf{A}_1$		

theoretical, whereas that of Theorem 1.2.2 depends essentially on the auxiliary material presented in §2 and §3.

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2. Preliminaries

2.1. **Finite quadratic forms.** A *finite quadratic form* is a finite abelian group $\mathcal{L}$ equipped with a map $q\colon \mathcal{L} \to \mathbb{Q}/2\mathbb{Z}$. satisfying $q(x+y) = q(x) + q(y) + 2b(x,y)$ and $q(nx) = n^2x$ for all $x, y \in \mathcal{L}$, where $b\colon \mathcal{L} \otimes \mathcal{L} \to \mathbb{Q}/\mathbb{Z}$ is a symmetric bilinear form (which is determined by q). To reduce the notation we write x^2 for $q(x)$ and $x \cdot y$ for $b(x,y)$. For a prime p, let $\mathcal{L}_p := \mathcal{L} \otimes \mathbb{Z}_p$, which is called the *p-primary part* of $\mathcal{L}$. Any finite quadratic form $\mathcal{L}$ can be written as an orthogonal sum of its p-primary components $\mathcal{L}_p$, *i.e.*, $\mathcal{L} = \bigoplus_p \mathcal{L}_p$ where the summation runs over all primes p. Denote by $\ell(\mathcal{L})$ the minimal number of generators of $\mathcal{L}$.

Consider a fraction $\frac{m}{n} \in \mathbb{Q}/2\mathbb{Z}$ with $g.c.d(m,n) = 1$ and $mn = 0 \bmod 2$. By $\langle \frac{m}{n} \rangle$, we denote the finite non-degenerate (see §2.2) quadratic form on $\mathbb{Z}/n\mathbb{Z}$ generated by an element of square $\frac{m}{n}$ and of order n. For an integer $k \geq 1$, let $\mathcal{U}(2^k)$ and $\mathcal{V}(2^k)$ be the quadratic forms on $\mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z}$ defined by the matrices

$$\mathcal{U}(2^k) = \begin{pmatrix} 0 & \frac{1}{2^k} \\ \frac{1}{2^k} & 0 \end{pmatrix} \qquad \text{and} \qquad \mathcal{V}(2^k) = \begin{pmatrix} \frac{1}{2^{k-1}} & \frac{1}{2^k} \\ \frac{1}{2^k} & \frac{1}{2^{k-1}} \end{pmatrix}.$$

Nikulin [22] showed that any finite quadratic form can be written as an orthogonal sum of cyclic summands of the form $\langle \frac{m}{n} \rangle$ and copies of $\mathcal{U}(2^k)$ and $\mathcal{V}(2^k)$.

The *Brown invariant* of a finite quadratic form $\mathcal{L}$ is the residue $\operatorname{Br}\mathcal{L} \in \mathbb{Z}/8\mathbb{Z}$ defined by the Gauss sum

$$\exp\left(\frac{1}{4} i\pi \operatorname{Br}\mathcal{L}\right) = |\mathcal{L}|^{-\frac{1}{2}} \sum_{i \in \mathcal{L}} \exp(i\pi x^2).$$

The Brown invariants of indecomposable p-primary blocks are as follows:

$\operatorname{Br}\langle \frac{2a}{p^{2s-1}} \rangle = 2(\frac{a}{p}) - (\frac{-1}{p}) - 1$, $\operatorname{Br}\langle \frac{2a}{p^{2s}} \rangle = 0$ (for p odd, $s \geq 1$ and g. c. d.$(a,p) = 1$),

$\operatorname{Br}\langle \frac{a}{2^k} \rangle = a + \frac{1}{2}k(a^2 - 1) \mod 8$ (for $k \geq 1$ and odd $a \in \mathbb{Z}$),

$\operatorname{Br}\mathcal{U}_{2^k} = 0$,

$\operatorname{Br}\mathcal{V}_{2^k} = 4k \mod 8$ (for all $k \geq 1$).

A finite quadratic form is called *even* if $x^2 = 0 \bmod \mathbb{Z}$ for all elements $x \in \mathcal{L}$ of order two; otherwise it is called *odd*. This definition implies that a quadratic form is odd if and only if it contains $\langle \pm\frac{1}{2} \rangle$ as an orthogonal summand.

2.2. **Integral lattices and discriminant forms.** An *(integral) lattice* is a free abelian group L of finite rank with a symmetric bilinear form $b\colon L \otimes L \to \mathbb{Z}$. For short, we use the multiplicative notation $x \cdot y$ for $b(x,y)$ and x^2 for $b(x,x)$. A lattice L is called *even* if a^2 is an even integer for all $a \in L$. It is called *odd* otherwise. The determinant $\det(L)$ is defined to be the determinant of the Gram matrix of b in any basis of L. Since the transition matrix between any two integral bases has determinant ± 1, $\det(L) \in \mathbb{Z}$ is well defined. A lattice L is called *non-degenerate* if $\det(L) \neq 0$; it is called *unimodular* if $\det(L) = \pm 1$.

From now on all lattices considered are even and non-degenerate.

Given a lattice L, the form $b : L \otimes L \to \mathbb{Z}$ can be extended by linearity to a form $(L \otimes \mathbb{Q}) \otimes_{\mathbb{Q}} (L \otimes \mathbb{Q}) \to \mathbb{Q}$. If L is non-degenerate, the dual group $L^* := \operatorname{Hom}(L, \mathbb{Z})$ can be identified with the subgroup

$$\{x \in L \otimes \mathbb{Q} \mid x \cdot y \in \mathbb{Z} \text{ for all } x \in L\}$$

Since the original bilinear form b on L is integer valued, L is a finite index subgroup of its dual. The quotient L^*/L is called the *discriminant group* of L and is denoted by $\mathcal{L}$ or $\operatorname{disc} L$. If $\{e_1, e_2, \dots e_n\}$ is a basis set for L and $\{e_1^*, e_2^*, \dots, e_n^*\}$ is the dual basis for L^*, then the Gram matrix $[e_i \cdot e_j]$ is exactly the matrix of the homomorphism $\varphi : L \to L^*$, $x \mapsto [y \mapsto x \cdot y]$. Hence one has $|\mathcal{L}| = |\det(L)|$. Note that $x \cdot y \in \mathbb{Z}$ whenever $x \in L$ or $y \in L$. Thus, $\mathcal{L}$ inherits from $L \otimes \mathbb{Q}$ a non-degenerate symmetric bilinear form $b_{\mathcal{L}} : \mathcal{L} \otimes \mathcal{L} \to \mathbb{Q}/\mathbb{Z}$; it is called the *discriminant form*. If L is even, this form $b_{\mathcal{L}}$ can be promoted to the quadratic extension $q_{\mathcal{L}} : \mathcal{L} \to \mathbb{Q}/2\mathbb{Z}$, $x \bmod L \mapsto x^2 \bmod 2\mathbb{Z}$. Hence, the discriminant form of an even lattice is a finite quadratic form. Accordingly, given a prime p, we use the notation $\operatorname{disc}_p L$ or $\mathcal{L}_p$ for the p-primary part of $\mathcal{L}$, *i.e.*, $\mathcal{L}_p = \mathcal{L} \otimes \mathbb{Z}_p$. Each discriminant group $\mathcal{L}$ decomposes into orthogonal sum $\mathcal{L} = \bigoplus_p \mathcal{L}_p$ of its p-primary components.

The *signature* of a non-degenerate lattice L is the pair (σ_+, σ_-) of its positive and negative inertia indices. Two non-degenerate integral lattices are said to have the same *genus* if their localizations over $\mathbb{R}$ and over $\mathbb{Q}_p$ are isomorphic. The following few statements give the relation between the genus of an even integral lattice and its discriminant form.

2.2.1. **Theorem** (Nikulin [22]). *The genus of an even integral lattice L is determined by its signature $(\sigma_+ L, \sigma_- L)$ and discriminant form* $\operatorname{disc} L$.

The existence of an even integral lattice L with a given signature is given by Nikulin's existence theorem (see Theorem 2.2.3).

2.2.2. **Theorem** (van der Blij [31]). *For any non-degenerate even integral lattice L one has* $\operatorname{Br} \mathcal{L} = \sigma_+ - \sigma_- \mod 8$.

We denote by $g(L)$ the set of all isomorphism classes of all non-degenerate even integral lattices with the same genus as L. Each set $g(L)$ is known to contain finitely many isomorphism classes.

Given a prime p, we define the *determinant* $\det_{\mathrm{p}}(\mathcal{L})$ as the determinant of the matrix of the quadratic from on $\mathcal{L}_p$ in an appropriate basis (see [21] and [22] for details). Unless $p = 2$ one has $\det_p(\mathcal{L}) = u/|\mathcal{L}_p|$ where u is a well defined element of $u \in \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2$. If $p = 2$, the determinant $\det_2(\mathcal{L})$ is well defined only if $\mathcal{L}_2$ is even.

2.2.3. **Theorem** (Nikulin [22]). *Let $\mathcal{L}$ be a finite quadratic form and let $\sigma_\pm$ be a pair of integers. Then, the following four conditions are necessary and sufficient for the existence of an even integral lattice L whose signature is (σ_+, σ_-) and whose discriminant form is $\mathcal{L}$:*

(1) $\sigma_\pm \geq 0$ *and* $\sigma_+ + \sigma_- \geq \ell(\mathcal{L})$;
(2) $\sigma_+ - \sigma_- = \operatorname{Br} \mathcal{L} \mod 8$;
(3) *for each* $p \neq 2$, *either* $\sigma_+ + \sigma_- > \ell_p(\mathcal{L})$ *or* $\det_p(\mathcal{L}) \equiv (-1)^{\sigma_-} \cdot |\mathcal{L}| \bmod (\mathbb{Z}_p^*)^2$;
(4) *either* $\sigma_+ + \sigma_- > \ell_2(\mathcal{L})$, *or* $\mathcal{L}_2$ *is odd, or* $\det_2(\mathcal{L}) \equiv \pm|\mathcal{L}| \bmod (\mathbb{Z}_2^*)^2$.

2.3. **Automorphisms of lattices.** An *isometry* of integral lattices is a homomorphism of abelian groups preserving the forms. The group of auto-isometries of L is denoted by $O(L)$. There is a natural homomorphism $d\colon O(L) \to \operatorname{Aut}(\mathcal{L})$, where $\operatorname{Aut}(\mathcal{L})$ denotes the group of automorphisms of $\mathcal{L}$ preserving the discriminant form q on $\mathcal{L}$. Obviously, one has $\operatorname{Aut}(\mathcal{L}) = \prod_p \operatorname{Aut}(\mathcal{L}_p)$, where the product runs over all primes. The restrictions of d to the p-primary components are denoted by $d_p : O(L) \to \operatorname{Aut}(\mathcal{L}_p)$.

Given a vector u in L with $u \neq 0$, the *reflection* against its orthogonal hyperplane is the automorphism

$$\begin{aligned} r_u :& L \to L \\ & x \mapsto x - 2\frac{(x \cdot u)}{u^2} u \end{aligned}$$

The reflection r_u is well-defined whenever $u \in (\frac{u^2}{2})L^*$. Note that $r_u^2 = \mathrm{id}$, *i.e.*, r_u is an involution. Each image $d_p(r_u) \in \operatorname{Aut}(\mathcal{L}_p)$ is also a reflection (see §2.6). If $u^2 = \pm 1$ or $u^2 = \pm 2$, then the induced automorphism $d(r_u)$ is the identity.

2.4. **Lattice extensions.** An even integral lattice L containing even lattice S called an *extension* of S. As *isomorphism* between two extensions $L_1 \supset S$ and $L_2 \supset S$ is an isometry between L_1 and L_2 taking S to S. In particular, if the isomorphism $L_1 \to L_2$ restricts to id on S, the extensions L_1 and L_2 are called *strictly isomorphic*. For a given subgroup A of O(S), we define A-*isomorphisms* of extensions of S as those which restrict to an element of A on S.

Recall that S is assumed non-degenerate, hence given a *finite index* extension $L \supset S$, one has $L \subset S^*$. Thus there are inclusions $S \subset L \subset L^* \subset S^*$ which imply $L/S \subset S^*/S = \mathcal{S}$. The subgroup $\mathcal{K} = L/S$ of $\mathcal{S}$ is called the *kernel* of the finite index extension $L \supset S$. Since L is an even integral lattice, the discriminant quadratic form on $\mathcal{S}$ restricts to zero on $\mathcal{K}$, *i.e.,* $\mathcal{K}$ is isotropic.

2.4.1. **Proposition** (Nikulin [22]). *Let S be a non-degenerate even lattice, and fix a subgroup $A \subset O(S)$. The map $L \mapsto \mathcal{K} = L/S \subset \mathcal{S}$ establishes a one-to-one correspondence between the set of A-isomorphism classes of finite index extensions $L \supset S$ and the set of A-orbits of isotropic subgroups $\mathcal{K} \subset \mathcal{S}$. Under this correspondence one has $L = \{x \in S^* \mid (x \bmod S) \in \mathcal{K}\}$ and $\mathcal{L} = \mathcal{K}^{\perp}/\mathcal{K}$.*

2.4.2. **Proposition** (Nikulin [22]). *Let $L \supset S$ be a finite index extension of a lattice S and let $\mathcal{K} \subset \mathcal{S}$ be its kernel. Then an auto-isometry $S \to S$ extends to L if and only if the induced automorphism of $\mathcal{S}$ preserves $\mathcal{K}$.*

An extension $L \supset S$ is called *primitive* if L/S is torsion free. Following Nikulin [22], we confine ourselves to the special case where L is unimodular. If S is a primitive non-degenerate sublattice of a unimodular lattice L then $S^{\perp}$ is also primitive in L and L is a finite index extension of $S \oplus S^{\perp}$. Furthermore, since $\operatorname{disc} L = 0$, the kernel $\mathcal{K} \subset \mathcal{S} \oplus \mathcal{S}^{\perp}$ is the graph of an anti-isometry $\psi\colon \mathcal{S} \to \operatorname{disc} S^{\perp}$. Hence the genus $g(S^{\perp})$ is determined by the genera $g(N)$ and $g(L)$. Conversely, given a lattice $N \in g(S^{\perp})$ and an anti-isometry $\psi\colon \mathcal{S} \to \mathcal{N}$, the graph of ψ is an isotropic subgroup $\mathcal{K} \subset \mathcal{S} \oplus \mathcal{S}^{\perp}$ and the corresponding finite index extension $S \oplus N \hookrightarrow L$ is a unimodular primitive extension of S with $S^{\perp} \cong N$. Note that an anti-isometry $\psi\colon \mathcal{S} \to \operatorname{disc} S^{\perp}$ induces a homomorphism $d^{\psi}: O(S) \to \operatorname{Aut}(\mathcal{N})$. Thus, since also an indefinite unimodular lattice is unique in its genus, we have the following theorem.

2.4.3. **Theorem** (Nikulin [22]). *Let L be an indefinite unimodular even lattice and $S \subset L$ a non-degenerate primitive sublattice. Fix a subgroup $A \subset O(S)$. Then the A-isomorphism class of a primitive extension $S \subset L$ is determined by*

(1) *a choice of a lattice $N \in g(S^\perp)$ and*
(2) *a choice of a double coset $c_N \in d^\psi(A)\backslash Aut(\mathcal{N})/\operatorname{Im} d$ (for a given N and some anti-isometry $\psi : \mathcal{S} \to \mathcal{N}$ inducing d^ψ).*

2.4.4. **Theorem** (Nikulin [22]). *Let L be an indefinite unimodular even lattice, $S \subset L$ a non-degenerate primitive sublattice and $\psi : \mathcal{S} \to \mathcal{N}$ the anti-isometry where, $N = S^\perp$. Then a pair of isometries $a_S \in O(S)$ and $a_N \in O(N)$ extends to L if and only if $d^\psi(a_S) = d(a_N)$.*

2.5. **Miranda–Morrison's theory.** Let p be a prime. Define

$$\Gamma_p := \{\pm 1\} \times \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2,$$
$$\Gamma_0 := \{\pm 1\} \times \{\pm 1\} \subset \{\pm 1\} \times \mathbb{Q}^\times/(\mathbb{Q}^\times)^2.$$

It is convenient to introduce the following subgroups related to Γ_p :

- $\Gamma_{p,0} := \{(1,1),(1,u_p),(-1,1),(-1,u_p)\} \subset \Gamma_p$; here, p is odd and u_p is the only nonzero element of $\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2$,
- $\Gamma_{2,0} := \{(1,1),(1,3),(1,5),(1,7),(-1,1),(-1,3),(-1,5),(-1,7)\} \subset \Gamma_2$,
- $\Gamma_p^{++} := \{1\} \times \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2 \subset \Gamma_{p,0}$,
- $\Gamma_{2,2} := \{(1,1),(1,5)\} \subset \Gamma_2^{++}$,
- $\Gamma'_{2,0} := \Gamma_{2,0}/\Gamma_{2,2}$ (and $\Gamma'_{p,0} := \Gamma_{p,0}$ for $p \neq 2$),
- $\Gamma_0^{--} := \{(1,1),(-1,-1)\} \subset \Gamma_0$.

Let, further,

$$\Gamma_{\mathbb{A},0} := \prod_p \Gamma_{p,0} \subset \Gamma_{\mathbb{A}} := \Gamma_{\mathbb{A},0} \cdot \sum_p \Gamma_p$$

where " $\cdot$ " denotes the sum of the subgroups. Note that

$$\Gamma_{\mathbb{A}} = \{(d_p, s_p) \in \prod_p \Gamma_p \,|(d_p, s_p) \in \Gamma_{p,0} \text{ for almost all } p\}$$

The natural map $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2 \to \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$ induces canonical maps

$$\varphi_p : \Gamma_0 \to \Gamma_{p,0}. \tag{2.1}$$

Let N be an indefinite lattice with $\operatorname{rk}(N) \geq 3$. We will use certain subgroups $\Sigma_p^\sharp(N) \subset \Gamma_{p,0}$ and $\Sigma_p(N) \subset \Gamma_p$. In the notation of [20] (which slightly differs from the notation in [18, 19]), one has $\Sigma_p^\sharp(N) := \Sigma^\sharp(N \otimes \mathbb{Z}_p)$ and $\Sigma_p(N) := \Sigma(N \otimes \mathbb{Z}_p)$; we refer the reader to [20] for the precise definitions. The subgroups $\Sigma_p^\sharp(N)$ are computed explicitly in [20] (see Theorem 12.1, 12.2, 12.3 and 12.4).

Also defined in [20] is the $\mathbb{F}_2$-module

$$E(N) := \Gamma_{\mathbb{A},0}/\prod_p \Sigma_p^\sharp(N) \cdot \Gamma_0. \tag{2.2}$$

This module is finite. Indeed, following [20], we call a prime p *regular* with respect to N if $\Sigma_p^\sharp(N) = \Gamma_{p,0}$. Crucial is the fact that a prime p is regular unless $p \mid \det(N)$; thus, (2.2) reduces to finitely many primes p:

$$E(N) = \Gamma_{\mathbb{A},0}/\prod_{p \mid \det(N)} \Sigma_p^\sharp(N) \cdot \Gamma_0. \tag{2.3}$$

2.5.1. **Theorem** (Miranda–Morrison [20]). *Let N be a non-degenerate indefinite even lattice with* $\mathrm{rk}(N) \geq 3$. *Then there is an exact sequence*

$$O(N) \xrightarrow{d} \mathrm{Aut}(\mathcal{N}) \xrightarrow{\mathrm{e}} E(N) \to g(N) \to 1, \tag{2.4}$$

where $g(N)$ is the genus group of N.

A simplified version of (2.3) computing the numeric invariants

$$e_p(N) := [\Gamma_{p,0} : \Sigma^\sharp(N)] \text{ and } \tilde{\Sigma}_p(N) := \varphi_p^{-1}(\Sigma_p^\sharp(N)) \subset \Gamma_0,$$

is found in [18, 19]. This gives us the size of the group $E(N)$: one has

$$|E(N)| = \frac{e(N)}{[\Gamma_0 : \tilde{\Sigma}(N)]} \tag{2.5}$$

where

$$e(N) := \prod_p e_p(N), \quad \tilde{\Sigma}(N) := \bigcap_p \tilde{\Sigma}_p(N),$$

and the product and intersection run over all primes p or, equivalently, over all primes $p \mid \det(N)$.

The following theorem can be deduced from Theorems 2.4.3 and 2.5.1.

2.5.2. **Theorem** (Miranda–Morrison [18, 19]). *Let S be a primitive sublattice of an even unimodular lattice L such that $N := S^\perp$ is a non-degenerate indefinite even lattice with* $\mathrm{rk}(N) \geq 3$. *Then the strict isomorphism classes of primitive extensions $S \hookrightarrow L$ are in a canonical one-to-one correspondence with the group $E(N)$.*

As explained §2.4, given a unimodular lattice L and a primitive sublattice $S \subset L$, one has an anti-isometry $\psi : \mathcal{S} \to \mathcal{N}$ (where $N = S^\perp$), which induces a homomorphism $d^\psi : O(S) \to \mathrm{Aut}(\mathcal{N})$. If N is indefinite and $\mathrm{rk}(N) \geq 3$, then $d(O(S)) \subset \mathrm{Aut}(\mathcal{N})$ is a normal subgroup with abelian quotient (see (2.4)) and we have a homomorphism $d^\perp : O(S) \to \mathrm{Aut}(\mathcal{N}) \xrightarrow{\mathrm{e}} E(N)$ independent of the choice of an anti-isometry ψ. The next statement follows from Theorems 2.5.1 and 2.4.3.

2.5.3. **Corollary.** *Let S be a primitive sublattice of an even unimodular lattice L such that $N := S^\perp$ is non-degenerate indefinite even lattice with $rk(N) \geq 3$ and let $A \subset O(S)$ be a subgroup. Then, the A-isomorphism classes of primitive extensions $S \hookrightarrow L$ are in a one-to-one correspondence with the $\mathbb{F}_2$-module* $\operatorname{coker} d^\perp(A)$.

2.6. **Reflections.** Recall that $\mathrm{Aut}(\mathcal{N}) = \prod_p \mathrm{Aut}(\mathcal{N}_p)$ where p runs over all primes. Let s be a prime and $\alpha \in \mathcal{N}_s$ such that

$$s^k\alpha = 0 \text{ and } \alpha^2 = \frac{2u}{s^k} \bmod 2\mathbb{Z}, \text{ g.c.d}(u, s) = 1, \, k \in \mathbb{N}. \tag{2.6}$$

We denote by $\mathcal{N}_s^\dagger$ the set of all elements $\alpha \in \mathcal{N}_s$ satisfying (2.6) and let $\mathcal{N}^\dagger = \bigcup_s \mathcal{N}_s^\dagger$. Then one can define a map,

$$\mathcal{N}_s \to \mathbb{Z}/s^k, \quad x \mapsto \frac{2(x \cdot \alpha)}{\alpha^2} \bmod s^k.$$

where $\alpha \in \mathcal{N}_s^\dagger$. Thus, there is a reflection $r_\alpha \in \mathrm{Aut}\,\mathcal{N}_s$ given by

$$r_\alpha : x \mapsto x - \frac{2(x \cdot \alpha)}{\alpha^2}\alpha.$$

If $\alpha^2 = \frac{1}{2} \mod \mathbb{Z}$ and $2\alpha = 0$ then $r_\alpha = \mathrm{id}$.

Let p be a prime and consider the homomorphism

$$\mathrm{Aut}(\mathcal{N}) = \prod_p \mathrm{Aut}(\mathcal{N}_p) \xrightarrow{\phi} \prod_p \Sigma_p(N)/\Sigma_p^\sharp(N)$$

which is the product of the epimorphisms

$$\phi_p : \mathrm{Aut}(\mathcal{N}_p) \twoheadrightarrow \Sigma_p(N)/\Sigma_p^\sharp(N)$$

introduced in Miranda–Morrison [20]. The images of the homomorphism ϕ_p can be computed on reflections as follows: For a prime s and an element $\alpha \in \mathcal{N}_s^\dagger$, the image of the reflection $r_\alpha \in \mathrm{Aut}(\mathcal{N}_s)$ under ϕ_s is given by $\phi_s(r_\alpha) = (-1, us^k)$, see (2.6). If $s = 2$ and $\alpha^2 = 0 \mod \mathbb{Z}$, then $\phi_s(r_\alpha)$ is only well-defined $\mod \Gamma_2^{++}$. If $s = 2$ and $\alpha^2 = \frac{1}{2} \mod \mathbb{Z}$, then $\phi_s(r_\alpha)$ is well-defined $\mod \Gamma_{2,2}$. In these cases to determine the value of $\phi_s(r_\alpha)$, we need more information about α and N.

Given another prime p, we define the *p-norm* $|\alpha|_p \in \{\pm 1\}$ of $\alpha \in \mathcal{N}_s^\dagger$ by

$$|\alpha|_p := \begin{cases} \chi_p(s^k) & \text{if } s \neq p, \\ \chi_p(u) & \text{if } s = p, \end{cases}$$

where the homomorphism $\chi_p : \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2 \to \{\pm 1\}$ is defined as

$$\chi_p(u) := \begin{cases} \left(\frac{u}{p}\right) & \text{if } p \neq 2, \\ u \bmod 4 & \text{if } p = 2. \end{cases}$$

Note that $|\alpha|_2$ is undefined when $p = 2$ and $\alpha^2 = 0 \mod \mathbb{Z}$. Following [1], given primes p and s and a vector $\alpha \in \mathcal{N}_s^\dagger$, we introduce the group

$$E_p(N) := \begin{cases} \{\pm 1\} & \text{if } p = 1 \bmod 4 \text{ and } e_p(N) \cdot |\tilde{\Sigma}_p(N)| = 8, \\ 1 & \text{otherwise,} \end{cases}$$

the map $\bar{\phi}_p : \mathcal{N}_s^\dagger \to E_p(N)$,

$$\bar{\phi}_p(\alpha) := \begin{cases} 1 & \text{if } E_p(N) = 1, \\ |\alpha|_p & \text{otherwise,} \end{cases}$$

and the map $\bar{\beta}_p : \mathcal{N}_s^\dagger \to \Gamma_0$,

$$\bar{\beta}_p(\alpha) := \begin{cases} (\delta_p(\alpha) \cdot |\alpha|_p, 1) & \text{if } p = 1 \bmod 4, \\ \delta_p(\alpha) \times |\alpha|_p & \text{otherwise,} \end{cases}$$

where the map

$$\delta_p(\alpha) := (-1)^{\delta_{p,s}}$$

(here $\delta_{p,s}$ is the conventional Kronecker symbol). Note that we have the assignment

$$r_\alpha \mapsto (\delta_p(\alpha), |\alpha|_p) \in \Gamma'_{p,0}.$$

The following lemmas provide an explicit description for the group $E(N)$ and compute the image of the homomorphism e on the reflections r_α for the special case when N has one or two irregular primes.

2.6.1. **Lemma** (Akyol–Degtyarev [1]). *Let N be a non-degenerate indefinite even lattice with $rk(N) \geq 3$, $\Sigma_2^\sharp(N) \supset \Gamma_{2,2}$, and assume that N has one irregular prime p. Then $E(N) = E_p(N)$ and $\mathrm{e}(r_\alpha) = \bar{\phi}_p(\alpha)$ for any $\alpha \in \mathcal{N}^\dagger$.*

2.6.2. **Lemma** (Akyol–Degtyarev [1]). *Let N be a non-degenerate indefinite even lattice with $rk(N) \geq 3$, $\Sigma_2^\sharp(N) \supset \Gamma_{2,2}$, and assume that N has two irregular primes p, q. Then*

$$E(N) = E_p(N) \times E_q(N) \times (\Gamma_0/\tilde{\Sigma}_p(N) \cdot \tilde{\Sigma}_q(N)),$$
$$\mathrm{e}(r_\alpha) = \bar{\phi}_p(\alpha) \times \bar{\phi}_q(\alpha) \times (\bar{\beta}_p(\alpha) \cdot \bar{\beta}_q(\alpha)),$$

for any $\alpha \in \mathcal{N}^\dagger$ such that $\alpha^2 \neq 0 \bmod \mathbb{Z}$ if $p = 2$ or $q = 2$.

2.6.3. **Corollary** (Akyol–Degtyarev [1]). *Under the hypothesis of Lemma 2.6.2, assume, in addition, that $|E(N)| = |E_p(N)| = 2$. Then $E(N) = E_p(N)$ and $\mathrm{e}(r_\alpha) = |\alpha|_p$ for any $\alpha \in \mathcal{N}^\dagger$.*

2.7. **Positive sign structure.** Let N be a non-degenerate lattice. The orthogonal projection of any positive definite 2-subspace in $N \otimes \mathbb{R}$ to any other such subspace is an isomorphism of vector spaces. Thus a choice of an orientation of one maximal positive definite subspace in $N \otimes \mathbb{R}$ defines a coherent orientation of any other. A choice of an orientation of a maximal positive definite subspace of $N \otimes \mathbb{R}$ is called a *positive sign structure*. We denote by $O^+(N)$ the subgroup of $O(N)$ consisting of the isometries preserving a positive sign structure. Obviously either $O^+(N) = O(N)$ or $O^+(N) \subset O(N)$ is a subgroup of index 2. In the latter case, each element of $O(N) \smallsetminus O^+(N)$ is called a $+$-*disorienting* isometry of N. Following [20], we define the map $\det_+ : O(N) \to \{\pm 1\}$ as

$$\det_+(a) := \begin{cases} -1 & \text{if } a \text{ preserves the positive sign structure,} \\ +1 & \text{if } a \text{ reserves the positive sign structure.} \end{cases}$$

Note that $\mathrm{Ker}(\det_+) = O^+(N)$.

2.7.1. **Proposition** (Miranda–Morrison [20]). *Let N be a non-degenerate indefinite even lattice with $\mathrm{rk}(N) \geq 3$. Then one has $\tilde{\Sigma}(N) \subset \Gamma_0^{--}$ if and only if $\det_+(a) = 1$ for all $a \in \mathrm{Ker}[d\colon O(N) \to \mathrm{Aut}(\mathcal{N})]$.*

The following lemma computes the images of the function $\det_+$ on reflections.

2.7.2. **Lemma** (Akyol–Degtyarev [1]). *Let N be a non-degenerate indefinite even lattice with $\mathrm{rk}(N) \geq 3$, $\Sigma_2^\sharp(N) \supset \Gamma_{2,2}$, and assume that there is a prime p such that $\tilde{\Sigma}_p(N) \subset \Gamma_0^{--}$. Then, for an element $\alpha \in \mathcal{N}^\dagger$ such that $r_\alpha \in \mathrm{Im}\, d$ and $\alpha^2 \neq 0 \bmod \mathbb{Z}$ if $p = 2$, one has $\det_+(r_\alpha) = \delta_p(\alpha) \cdot |\alpha|_p$.*

Defined in [19], we introduce the group

$$E^+(N) := \Gamma_{\mathbb{A},0}/\prod_p \Sigma_p^\sharp(N) \cdot \Gamma_0^{--}. \tag{2.7}$$

(Similar to (2.2) and (2.3) the actual computation reduces to finitely many primes $p \mid \det(N)$.) As in Theorem 2.5.1 there is an exact sequence

$$O^+(N) \xrightarrow{d} \mathrm{Aut}(\mathcal{N}) \xrightarrow{\mathrm{e}^+} E^+(N) \to g(N) \to 1.$$

The size of the group $E^+(N)$ is also computed in [19]: one replaces $[\Gamma_0 : \tilde{\Sigma}(N)]$ in (2.5) with $[\Gamma_0^{--} : \tilde{\Sigma}(N) \cap \Gamma_0^{--}]$. For an irregular prime p, we denote $\tilde{\Sigma}_p^+(N) := \tilde{\Sigma}_p(N) \cap \Gamma_0^{--}$.

Given a unimodular even lattice L and a primitive sublattice $S \subset L$ such that $N := S^\perp$ is a non-degenerate indefinite lattice with $\mathrm{rk}(N) \geq 3$, we have a well-defined homomorphism $d_+^\perp \colon O(S) \to E^+(N)$, *cf.* the definition of $d^\perp$ in §2.5.

Let p and s be two irregular primes and choose an element $\alpha \in \mathcal{N}_s^\dagger$ as in (2.6), we introduce the group

$$E_p^+(N) := \begin{cases} E_p(N) & \text{if } p = 1 \bmod 4, \\ \Gamma_0/\tilde{\Sigma}_p(N) \cdot \Gamma_0^{--} & \text{otherwise,} \end{cases}$$

the map $\bar{\phi}_p^+ : \mathcal{N}_s^\dagger \to E_p^+(N)$,

$$\bar{\phi}_p^+(\alpha) := \begin{cases} \bar{\phi}_p(\alpha) & \text{if } p = 1 \bmod 4, \\ \delta_p(\alpha) \cdot |\alpha|_p & \text{if } p \neq 1 \bmod 4 \text{ and } E_p^+(N) \neq 1, \\ 1 & \text{if } p \neq 1 \bmod 4 \text{ and } E_p^+(N) = 1, \end{cases}$$

and the map $\bar{\beta}_p^+ : \mathcal{N}_s^\dagger \to \Gamma_0^{--}$,

$$\bar{\beta}_p^+(\alpha) := \begin{cases} \delta_p(\alpha) \cdot |\alpha|_p & \text{if } p = 1 \bmod 4, \\ |\alpha|_p & \text{if } p \neq 1 \bmod 4 \text{ and } E_p^+(N) \neq 1, \\ \mathrm{proj}(\bar{\beta}_p(\alpha)) & \text{if } p \neq 1 \bmod 4 \text{ and } E_p^+(N) = 1, \end{cases}$$

where $\mathrm{proj} : \Gamma_0 \to \Gamma_0/\tilde{\Sigma}_p(N) = \Gamma_0^{--}$ is the projection map. Next lemma computes the group $E^+(N)$ and the values of the homomorphism e^+ on the reflections r_α

2.7.3. **Lemma** (Akyol–Degtyarev [1]). *Let N be a non-degenerate indefinite even lattice with $\mathrm{rk}(N) \geq 3$, $\Sigma_2^\sharp(N) \supset \Gamma_{2,2}$ and assume that N has two irregular primes p, q. Then*

$$E^+(N) = E_p^+(N) \times E_q^+(N) \times (\Gamma_0^{--}/\tilde{\Sigma}_p^+(N) \cdot \tilde{\Sigma}_q^+(N))$$
$$\mathrm{e}^+(r_\alpha) = \bar{\phi}_p^+(\alpha) \times \bar{\phi}_q^+(\alpha) \times (\bar{\beta}_p^+(\alpha) \cdot \bar{\beta}_q^+(\alpha))$$

for any $\alpha \in \mathcal{N}^\dagger$ such that $\alpha^2 \neq 0 \bmod \mathbb{Z}$ if $p = 2$ or $q = 2$.

2.8. **Root Systems.** A *root* in a lattice L is an element $v \in L$ of square -2. A *root system* is a negative definite lattice generated by its roots. Each root system splits uniquely into orthogonal sum of its irreducible components. As explained in [2], the irreducible root systems are $\mathbf{A}_n, n \geq 1$, $\mathbf{D}_m, m \geq 4$ and $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$. The corresponding discriminant forms are as follows:

$$\operatorname{disc} \mathbf{A}_n = \left\langle -\frac{n}{n+1} \right\rangle, \quad \operatorname{disc} \mathbf{D}_{2k+1} = \left\langle -\frac{2k+1}{4} \right\rangle,$$
$$\operatorname{disc} \mathbf{D}_{8k\pm 2} = 2\left\langle \mp\frac{1}{2} \right\rangle, \quad \operatorname{disc} \mathbf{D}_{8k} = \mathcal{U}(2), \quad \operatorname{disc} \mathbf{D}_{8k+4} = \mathcal{V}(2),$$
$$\operatorname{disc} \mathbf{E}_6 = \left\langle \frac{2}{3} \right\rangle, \quad \operatorname{disc} \mathbf{E}_7 = \left\langle \frac{1}{2} \right\rangle, \quad \operatorname{disc} \mathbf{E}_7 = 0.$$

Given a root system S, the group generated by reflections (defined by the roots of S) acts simply transitively on the set of Weyl chambers of S. The roots constituting a single Weyl chamber form a *standard basis* for S; these roots are naturally identified with the vertices of the Dynkin graph $\Gamma := \Gamma_S$ Thus, ones has an obvious homomorphism

$$\mathrm{Sym}(\Gamma) \to O(S) \to \mathrm{Aut}(\mathcal{S})$$

where $\mathrm{Sym}(\Gamma)$ denotes the symmetries of Γ. Irreducible root systems correspond to connected Dynkin graphs. The following statement follows immediately from the classification of connected Dynkin graphs (see N. Bourbaki [2]).

2.8.1. **Lemma.** *Let $\Gamma = \Gamma_S$ be the connected Dynkin graph of an irreducible root system S. Then,*

(1) *if S is $\mathbf{A}_1$, $\mathbf{E}_7$ or $\mathbf{E}_8$, then* $\mathrm{Sym}(\Gamma) = 1$
(2) *if S is $\mathbf{D}_4$, then* $\mathrm{Sym}(\Gamma) = \mathbb{S}_3$
(3) *for all other types,* $\mathrm{Sym}(\Gamma) = \mathbb{Z}_2$

If S is $\mathbf{A}_p$, $p \geq 2$, $\mathbf{D}_{2k+1}$ or $\mathbf{E}_8$, then the only nontrivial symmetry of Γ induces $-\,\mathrm{id}$ on $\mathcal{S}$. If S is $\mathbf{E}_8$ then $\mathcal{S} = 0$ and if S is $\mathbf{A}_1$, $\mathbf{A}_7$ of $\mathbf{D}_{2k}$, the groups $\mathcal{S}$ are $\mathbb{F}_2$ modules and $-\,\mathrm{id} = \mathrm{id}$ on $\mathrm{Aut}\,\mathcal{S}$.

Further details on irreducible root systems are found in N. Bourbaki [2].

3. Simple quartics

3.1. **Quartics and $K3$-surfaces.** A *quartic* is a surface $X \subset \mathbb{P}^3$ of degree four. A quartic is *simple* if all its singular points are simple, i.e., those of type $\mathbf{A}, \mathbf{D}, \mathbf{E}$. Isomorphism classes of simple singularities are known to be in a one-to-one correspondence with those of irreducible root systems (see Dufree [10] for details). Hence, a set of simple singularities can be identified with a root system, the irreducible summands of the latter (see §2.8) correspond to the individual singularity points.

Let $X \subset \mathbb{P}^3$ be a simple quartic and consider its minimal resolution of singularities $\tilde{X}$. It is well known that $\tilde{X}$ is a $K3$-surface; hence, $H_2(\tilde{X}) \cong 2\mathbf{E}_8 \oplus 3\mathbf{U}$, where $\mathbf{U}$ is the hyperbolic plane defined as $\mathbf{U} := \mathbb{Z}u_1 \oplus \mathbb{Z}u_2$, $u_1^2 = u_2^2 = 0$ and $u_1 \cdot u_2 = 1$. Note that $2\mathbf{E}_8 \oplus 3\mathbf{U}$ is the only even unimodular lattice of signature $(\sigma_+, \sigma_-) = (3, 19)$. We fix the notation $\mathbf{L}_X := H_2(\tilde{X})$ and $\mathbf{L} := 2\mathbf{E}_8 \oplus 3\mathbf{U}$.

For each simple singular point p of X the components of the exceptional divisor are smooth rational (-2)-curves spanning a root lattice in $\mathbf{L}_X$. These sublattices are obviously orthogonal and their orthogonal sum, identified with the set of singularities of X, is denoted by $\mathbf{S}_X$. The rank $\mathrm{rk}(\mathbf{S}_X)$ equals the total Milnor number $\mu(X)$. Since $\sigma_-(\mathbf{L}) = 19$ and $\mathbf{S}_X \subset \mathbf{L}$ is negative definite, one has $\mu(X) \leq 19$ (see [29], *cf.*, [24]). If $\mu(X) = 19$, the quartic is called *maximizing*. We introduce the following objects:

- $\mathbf{S}_X \subset \mathbf{L}_X$: the sublattice generated the set of classes of exceptional divisors contracted by the blow-up map $\tilde{X} \to X$;
- $h_X \in \mathbf{L}_X$: the class of the pull-back of a generic plane section of X;
- $\mathbf{S}_{X,h} = \mathbf{S}_X \oplus \mathbb{Z}h_X \subset \mathbf{L}_X$;
- $\tilde{\mathbf{S}}_X \subset \tilde{\mathbf{S}}_{X,h} \subset \mathbf{L}_X$: the primitive hulls of $\mathbf{S}_X$ and $\mathbf{S}_{X,h}$, respectively, *i.e*, $\tilde{\mathbf{S}}_X := (\mathbf{S}_X \otimes \mathbb{Q}) \cap \mathbf{L}_X$ and $\tilde{\mathbf{S}}_{X,h} := (\mathbf{S}_{X,h} \otimes \mathbb{Q}) \cap \mathbf{L}_X$, .
- $\omega_X \subset \mathbf{L}_X \otimes \mathbb{R}$: the oriented 2-subspace spanned by the real and imaginary parts of the class of a holomorphic 2-form on $\tilde{X}$ (the *period* of $\tilde{X}$).

The triple $(\mathbf{S}_X, h_X, \mathbf{L}_X)$ is called the *homological type of* X.

3.2. **Abstract homological types.** As explained above the set of singularities of a quartic $X \in \mathbb{P}^3$ can be viewed as a root lattice $\mathbf{S} \subset \mathbf{L}$.

3.2.1. **Definition.** A *configuration* (extending a given set of singularities $\mathbf{S}$) is a finite index extension $\tilde{\mathbf{S}}_h \supset \mathbf{S}_h := \mathbf{S} \oplus \mathbb{Z}h$, $h^2 = 4$, satisfying the following conditions:

(1) each root $r \in (\mathbf{S} \otimes \mathbb{Q}) \cap \tilde{\mathbf{S}}_h$ with $r^2 = -2$ is in $\mathbf{S}$,
(2) $\tilde{\mathbf{S}}_h$ does not contain an element v with $v^2 = 0$ and $v \cdot h = 2$.

An *automorphism* of a configuration $\tilde{\mathbf{S}}_h$ is an auto-isometry of $\tilde{\mathbf{S}}_h$ preserving h. The group of automorphisms of $\tilde{\mathbf{S}}_h$ is denoted by $\mathrm{Aut}_h(\tilde{\mathbf{S}}_h)$. One has the obvious inclusions $\mathrm{Aut}_h(\tilde{\mathbf{S}}_h) \subset O(\tilde{\mathbf{S}}) \subset O(\mathbf{S})$, the latter is due to (1) in Definition 3.2.1, since $\mathbf{S}$ is recovered as the sublattice in $h^\perp \subset \tilde{\mathbf{S}}_h$ generated by roots.

3.2.2. **Definition.** An *abstract homological type* extending a fixed set of singularities $\mathbf{S}$ is an extension of $\mathbf{S}_h := \mathbf{S} \oplus \mathbb{Z}h$, $h^2 = 4$, to a lattice L isomorphic to $2\mathbf{E}_8 \oplus 3\mathbf{U}$, such that the primitive hull $\tilde{\mathbf{S}}_h$ of $\mathbf{S}_h$ in L is a configuration.

An abstract homological type is uniquely determined by the triple $\mathcal{H} = (\mathbf{S}, h, \mathbf{L})$. An *isomorphism* between two abstract homological types $\mathcal{H}_i = (\mathbf{S}_i, h_i, \mathbf{L}_i)$, $i = 1, 2$, is an isometry $\mathbf{L}_1 \to \mathbf{L}_2$, taking h_1 and $\mathbf{S}_1$ to h_2 and $\mathbf{S}_2$, respectively (as a set).

Given an abstract homological type $\mathcal{H} = (\mathbf{S}, h, \mathbf{L})$, we let $\tilde{\mathbf{S}} := (\mathbf{S} \otimes \mathbb{Q}) \cap \mathbf{L}$ and $\tilde{\mathbf{S}}_h := (\mathbf{S}_h \otimes \mathbb{Q}) \cap \mathbf{L}$ be the primitive hulls of $\mathbf{S}$ and $\mathbf{S}_h$, respectively. Note that $\tilde{\mathbf{S}} = h^\perp_{\tilde{\mathbf{S}}_h}$, *i.e.*, $\tilde{\mathbf{S}}$ is also the primitive hull of $h^\perp$. The orthogonal complement $\mathbf{S}_h^\perp$ is a non-degenerate lattice with $\sigma_+ \mathbf{S}^\perp = 2$. It follows that all positive definite 2-subspaces in $\mathbf{S}_h^\perp \otimes \mathbb{R}$ can be oriented in a coherent way (see §2.7).

3.2.3. **Definition.** An *orientation* of an abstract homological type $\mathcal{H} = (\mathbf{S}, h, \mathbf{L})$ is a choice θ of one of the coherent orientations of positive definite 2-subspaces of $\mathbf{S}_h^\perp \otimes \mathbb{R}$

An *isomorphism* between two oriented singular homological type $(\mathcal{H}_i, \theta_i)$, $i = 1, 2$, is an isomorphism $\mathcal{H}_1 \to \mathcal{H}_2$, taking θ_1 to θ_2. A singular homological type is called *symmetric* if $(\mathcal{H}, \theta)$ is isomorphic to $(\mathcal{H}, -\theta)$ for some orientation θ of $\mathcal{H}$, i.e., $\mathcal{H}$ admits an automorphism reversing the orientation.

3.3. **Classification of singular quartics.** Due to Saint-Donat [26] and Urabe [29], a triple $\mathcal{H} = (\mathbf{S}, h, \mathbf{L})$ is isomorphic to the homological type $(\mathbf{S}_X, h_X, \mathbf{L}_X)$ of a simple quartic $X \subset \mathbb{P}^3$ if and only if $\mathcal{H}$ is an abstract homological type in the sense of Definition 3.2.2. In this case, the oriented 2-subspace ω_X introduced in §3.1 defines an orientation of $\mathcal{H}$.

3.3.1. **Theorem** (see Theorem 2.3.1 in [7])**.** *The map sending a simple quartic surface $X \subset \mathbb{P}^3$ to its oriented homological type establishes a one to one correspondence between the set of equisingular deformation classes of quartics with a given set of simple singularities* $\mathbf{S}$ *and the set of isomorphism classes of oriented abstract homological types extending* $\mathbf{S}$. *Complex conjugate quartics have isomorphic homological types that differ by the orientations.*

3.3.2. **Definition.** A quartic X is called *non-special* if its homological type is primitive, *i.e.*, $\mathbf{S}_h \subset \mathbf{L}$ is a primitive sublattice.

Note that the homological type $\mathcal{H} = (\mathbf{S}, h, \mathbf{L})$ is primitive if and only if $\tilde{\mathbf{S}}_h = \mathbf{S}_h$, in this case, one has $\operatorname{disc} \tilde{\mathbf{S}}_h = \mathcal{S} \oplus \langle \frac{1}{4} \rangle$ and $\mathrm{Aut}_h(\tilde{\mathbf{S}}_h) = O(\mathbf{S})$.

For a given set of simple singularities $\mathbf{S}$, the corresponding equisingular stratum of quartics is denoted by $\mathcal{M}(\mathbf{S})$. Our primary interest is the family $\mathcal{M}_1(\mathbf{S}) \subset \mathcal{M}(\mathbf{S})$ constituted by the non-special quartics with the set of singularities $\mathbf{S}$. More generally, since the kernel $\mathcal{K}$ of the finite index extension $\mathbf{S}_h \subset \tilde{\mathbf{S}}_h$ is obviously invariant under equisingular deformations, one can consider the strata $\mathcal{M}_*(\mathbf{S}) \subset \mathcal{M}(\mathbf{S})$ where the subscript $*$ is the sequence of invariant factors of the kernel $\mathcal{K}$.

4. Proofs

4.1. **Proof of Theorem 1.2.1.** Note that $X \smallsetminus (\operatorname{Sing} X \cup H) \cong \tilde{X} \smallsetminus (E \cup H)$, where $\tilde{X}$ is the minimal resolution of X and E is the exceptional divisor of the blow up $\tilde{X} \to X$. Recall that $\mathbf{S}_X$ is the sublattice in $\mathbf{L}_X = H_2(\tilde{X})$ generated by the components of E (see §3.1). Thus, one has $H_2(E \cup H) = \mathbf{S}_X \oplus \mathbb{Z} h_X \subset \mathbf{L}_X = H_2(\tilde{X})$.

We have the following cohomology exact sequence of pair $(\tilde{X}, E \cup H)$:

$$\cdots \xrightarrow{j^*} H^2(\tilde{X}) \xrightarrow{i^*} H^2(E \cup H) \xrightarrow{\delta} H^3(\tilde{X}, E \cup H) \xrightarrow{j^*} \underbrace{H^3(\tilde{X})}_{0} \to \cdots .$$

Hence, $H^3(\tilde{X}, E \cup H) = \operatorname{coker} i^*$. By universal coefficients, since all groups involved are free, i^* is the adjoint of the map

$$i_* : H_2(E \cup H) \to H_2(X),$$

which is the inclusion $\mathbf{S}_{X,h} \hookrightarrow \mathbf{L}_X$. Thus, we have an exact sequence

$$0 \to H_2(\tilde{X}, E \cup H) \xrightarrow{i_*} H_2(\tilde{X}) \to \tilde{\mathbf{S}}_{X,h} / \mathbf{S}_{X,h} \oplus \mathrm{F} \to 0,$$

where F is a finitely generated free abelian group. This sequence can be regarded as a free resolution of $\tilde{\mathbf{S}}_{X,h} / \mathbf{S}_{X,h} \oplus \mathrm{F}$ and, by the definition of derived functor, we have the following isomorphisms

$$\operatorname{coker} i^* = \operatorname{Ext}\left(\tilde{\mathbf{S}}_{X,h} / \mathbf{S}_{X,h} \oplus \mathrm{F}, \mathbb{Z}\right) = \operatorname{Ext}\left(\tilde{\mathbf{S}}_{X,h} / \mathbf{S}_{X,h}, \mathbb{Z}\right).$$

Combining these observations with Poincaré–Lefschetz duality $H_1(\tilde{X} \smallsetminus (E \cup H)) = H^3(\tilde{X}, E \cup H)$, we conclude that

$$H_1(\tilde{X} \smallsetminus (E \cup H)) = \operatorname{Ext}\left(\tilde{\mathbf{S}}_{X,h} / \mathbf{S}_{X,h}, \mathbb{Z}\right) \cong \tilde{\mathbf{S}}_{X,h} / \mathbf{S}_{X,h}$$

(the last isomorphism being not natural). In particular $H_1(\tilde{X} \smallsetminus (E \cup H)) = 0$ if and only if $\mathbf{S}_{X,h} = \tilde{\mathbf{S}}_{X,h}$ *i.e*, if and only if X is non-special. □

4.2. **Proof of Theorem 1.2.2.** For the reader's convenience, we divide the proof into three propositions; Theorem 1.2.2 is their immediate consequence.

4.2.1. **Proposition.** *Realizable are all sets of singularities that can be obtained by a perturbation from either the* 59 *maximizing sets of singularities listed in Table* 1 *or* 19 *sets of singularities with the Milnor number* 18 *listed in Table* 2.

Proof. According to Theorems 3.3.1 and Definition 3.3.2, a set of singularities $\mathbf{S}$ is realized by a non-special quartic if and only if $\mathbf{S}$ extends to a primitive homological type. Thus, we are interested in primitive extensions $\mathbf{S}_h \hookrightarrow \mathbf{L} = 3\mathbf{U} \oplus 3\mathbf{E}_8$. Since the homological type is primitive, one has $\operatorname{disc} \tilde{\mathbf{S}}_h = \mathcal{S} \oplus \langle \frac{1}{4} \rangle$, and the realizable sets are easily found by using Nikulin's Existence Theorem (Theorem 2.2.3) applied to the genus of the transcendental lattice $\mathbf{T} := \mathbf{S}^\perp$, which is determined by $\mathbf{S}$, see §2.4. Implementing the algorithm in GAP [12], we found that 2872 sets of simple singularities are realized by non-maximal non-special quartics and 59 sets of simple singularities are realized by maximal non-special quartics. According to E. Looijenga [17], deformation classes of perturbations of an individual simple singularity of type $\mathbf{S}$ are in a one-to-one correspondence with the isomorphism classes of primitive extensions $\mathbf{S}' \hookrightarrow \mathbf{S}$ of root lattices, see §2.8 and §2.4. As shown in [11], the latter is the case if and only if the Dynkin graph of $\mathbf{S}'$ is an induced subgraph of that of $\mathbf{S}$. Hence, given a simple quartic X, any perturbation X to a simple quartic X' gives rises to a perturbation of the set of singularities $\mathbf{S}$ of X to the set of singularities $\mathbf{S}'$ of X'. Conversely, any induced subgraph of the Dynkin graph of a simple quartic X is that of an appropriate small perturbation X' of X. Proof of this statement repeats, almost literally, the proof of a similar theorem for plane sextic curves (see Proposition 5.1.1 in [6]). Accordingly, the list of 2872 sets of simple singularities realized by non-maximal non-special quartics is compared against the list of all perturbations of the 59 maximizing sets of singularities given in Table 1 and 19 sets of singularities with Milnor number 18 given in Table 2. The two lists coincide. □

Let $\mathbf{S}$ be one of the realizable sets of singularities and $\mathbf{T}$ a representative of the genus $g(\mathbf{S}_h^\perp)$. By Theorem 3.3.1, the connected components of the space $\mathcal{M}_1(\mathbf{S})$ modulo complex conjugation $\operatorname{conj} : \mathbb{P}^3 \to \mathbb{P}^3$ are enumerated by the isomorphism classes of primitive homological types extending $\mathbf{S}$. We investigate these isomorphism classes separately for the maximizing case, *i.e.*, $\mu(\mathbf{S}) = 19$, and non-maximizing case, *i.e*, $\mu(\mathbf{S}) \leq 18$.

If $\mu(\mathbf{S}) = 19$, the transcendental lattice $\mathbf{T}$ is a positive definite sublattice of rank 2, and the numbers (r, c) of connected components of the space $\mathcal{M}_1(\mathbf{S})$ listed in Table 1 can easily be computed by Gauss theory of binary quadratic forms [13] (A. Degtyarev, private communication); details will appear elsewhere. Thus, throughout the rest of the proof we assume $\mu(\mathbf{S}) \leq 18$.

4.2.2. **Proposition.** *For each realizable set of singularities* $\mathbf{S}$ *with* $\mu(\mathbf{S}) \leq 18$*, the space* $\mathcal{M}_1(\mathbf{S})/\operatorname{conj}$ *is connected.*

Proof. If $\mu(\mathbf{S}) \leq 18$, then $\mathbf{T}$ is an indefinite lattice with $\operatorname{rk}\mathbf{T} \geq 3$ and we can apply Mirranda–Morison's theory. We try to enumerate primitive homological types $\mathcal{H} = (\mathbf{S}, h, \mathbf{L})$ extending $\mathbf{S}$, *i.e.*, the primitive extensions $\mathbf{S}_h \hookrightarrow L$. Since the extension is primitive, $\tilde{\mathbf{S}}_h = \mathbf{S}_h$, one has $\operatorname{disc}\tilde{\mathbf{S}}_h = \mathcal{S} \oplus \langle \frac{1}{4} \rangle$ and $\operatorname{Aut}_h(\mathbf{S}_h) \cong O(\mathbf{S})$. Then we have a well-defined homomorphism $d^\perp \colon O(\mathbf{S}) \to E(\mathbf{T})$, and by Corollary 2.5.3,

$$\pi_0(\mathcal{M}_1(\mathbf{S})/\operatorname{conj}) \cong \operatorname{Coker}\left(d^\perp : O(\mathbf{S}) \to E(\mathbf{T})\right). \tag{4.1}$$

Thus, the space $\mathcal{M}_1(\mathbf{S})/\operatorname{conj}$ is connected (equivalently, the primitive homological type extending $\mathbf{S}$ is unique up to isomorphism) if and only if the map $d^\perp$ is surjective, and it is this latter statement that we prove below.

Out of the 2872 sets of singularities realized by non-special non-maximizing quartics, for 2830 sets of singularities one gets $E(\mathbf{T}) = 1$ by using (2.5), and the assertion follows automatically.

For the remaining 42 cases, one has $|E(\mathbf{T})| \neq 1$. Among these, there are 18 set of singularities containing a point of type $\mathbf{A}_4$ and satisfying the hypothesis of Lemma 2.6.1 or Corollary 2.6.3 with $p = 5$. For these set of singularities one has $|E(\mathbf{T})| = 2$ and a nontrivial symmetry of any type $\mathbf{A}_4$ point maps to the generator $-1 \in E(\mathbf{T})$.

There are 8 sets of singularities containing a point of type $\mathbf{A}_2$ and satisfying the hypothesis of Lemma 2.6.2 with $p = 2$, $q = 3$. For these 8 cases, one has $|E(\mathbf{T})| = 2$ and a nontrivial symmetry of any type $\mathbf{A}_2$ point maps to the generator $-1 \in E(\mathbf{T})$.

Table 3. Extremal singularities

Singularities	(p,q)	$\mathrm{e}\colon \operatorname{Aut}(\mathcal{T}) \to E(\mathbf{T})$	generators of $E(\mathbf{T})$
$\mathbf{E}_8 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_2$	$(2,3)$	$\mathrm{e}(r_\alpha) = \delta_2(\alpha)\cdot\delta_3(\alpha)\cdot\lvert\alpha\rvert_2\cdot\lvert\alpha\rvert_3$	$\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$2\mathbf{E}_6 \oplus 2\mathbf{A}_3$	$(2,3)$	$\mathrm{e}(r_\alpha) = \delta_2(\alpha)\cdot\delta_3(\alpha)\cdot\lvert\alpha\rvert_2\cdot\lvert\alpha\rvert_3$	symmetry of $\mathbf{A}_3$
$\mathbf{D}_{11} \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$	$(2,3)$	$\mathrm{e}(r_\alpha) = \delta_2(\alpha)\cdot\delta_3(\alpha)\cdot\lvert\alpha\rvert_2\cdot\lvert\alpha\rvert_3$	$\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$2\mathbf{D}_7 \oplus 2\mathbf{A}_2$	$(2,3)$	$\mathrm{e}(r_\alpha) = \delta_2(\alpha)\cdot\delta_3(\alpha)\cdot\lvert\alpha\rvert_2\cdot\lvert\alpha\rvert_3$	$\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$2\mathbf{D}_5 \oplus 2\mathbf{A}_4$	$(2,5)$	$\mathrm{e}(r_\alpha) = \delta_2(\alpha)\cdot\delta_5(\alpha)$	$\mathbf{A}_4 \leftrightarrow \mathbf{A}_4$
$\mathbf{D}_5 \oplus \mathbf{A}_6 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$	$(2,3)$	$\mathrm{e}(r_\alpha) = \delta_2(\alpha)\cdot\delta_3(\alpha)\cdot\lvert\alpha\rvert_2\cdot\lvert\alpha\rvert_3$	$\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$\mathbf{A}_7 \oplus \mathbf{A}_4 \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$	$(2,3)$	$\mathrm{e}(r_\alpha) = \delta_2(\alpha)\cdot\delta_3(\alpha)\cdot\lvert\alpha\rvert_2\cdot\lvert\alpha\rvert_3$	$\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$2\mathbf{A}_6 \oplus 2\mathbf{A}_3$	$(2,7)$	$\mathrm{e}(r_\alpha) = \lvert\alpha\rvert_2\cdot\lvert\alpha\rvert_7$	symmetry of $\mathbf{A}_3$
$2\mathbf{A}_6 \oplus 3\mathbf{A}_2$	$(3,7)$	$\mathrm{e}(r_\alpha) = \delta_3(\alpha)\cdot\delta_7(\alpha)\rvert\cdot\alpha\rvert_3\cdot\lvert\alpha\rvert_7$	$\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$

For the following 4 sets of singularities,

$$\mathbf{D}_9 \oplus \mathbf{A}_3 \oplus 3\mathbf{A}_2, \quad \mathbf{D}_7 \oplus \mathbf{D}_5 \oplus 3\mathbf{A}_2,$$
$$\mathbf{A}_{11} \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2, \quad \mathbf{A}_8 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_2,$$

one has $|E(\mathbf{T})| = 4$. Each of these 4 sets has two irregular primes $p = 2$, $q = 3$, and for all of them the homomorphism given by Lemma 2.6.2 is

$$\mathrm{e}(r_\alpha) = (\delta_2(\alpha)\cdot\delta_3(\alpha), |\alpha|_2\cdot|\alpha|_3) \in \{\pm 1\} \times \{\pm 1\}.$$

A symmetry of any type $\mathbf{A}_2$ point and a transposition $\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$ give rise to reflections $r_\alpha, r_\sigma \in \mathcal{T}$ with $\alpha^2 = \frac{2}{3}$ and $(\sigma)^2 = \frac{4}{3}$. The images $\mathrm{e}(r_\alpha) = (-1,-1)$ and $\mathrm{e}(r_\sigma) = (-1,1)$ are linearly independent, thus generating the group $E(\mathbf{T})$.

The 9 sets of singularities listed in Table 3 still satisfy the assumptions of Lemma 2.6.2, which yields $|E(\mathbf{T})| = 2$. Also shown in the table are the irregular primes (p,q), the homomorphism $\mathrm{e}\colon \operatorname{Aut}(\mathcal{T}) \to E(\mathbf{T})$, and an automorphism of $\mathbf{S}$ generating $E(\mathbf{T})$.

Finally, what remains are the three sets of singularities

$$\mathbf{D}_4 \oplus 2\mathbf{A}_4 \oplus 3\mathbf{A}_2, \quad 2\mathbf{A}_7 \oplus 2\mathbf{A}_2 \quad 2\mathbf{A}_4 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_2,$$

to which Lemmas 2.6.1, 2.6.2 or Corollary 2.6.3 do not apply. For them, we compute the group $E(\mathbf{T})$ directly from the definition (2.2) which can be restated as

$$E(T) = \prod_{p|\det(T)} \Gamma_{p,0} \Big/ \prod_{p|\det(T)} \Sigma_p^\sharp(T)\cdot\varphi(\Gamma_0),$$

where we identify the inclusion $\Gamma_0 \hookrightarrow \Gamma_{\mathbb{A},0}$ with the product $\varphi := \prod_p \varphi_p$ (see 2.1). For example, for the case

$$2\mathbf{A}_4 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_2,$$

the computation can be summarized in the following table:

	$\Gamma_{5,0}$		$\Gamma_{3,0}$		$\Gamma'_{2,0}$	
generator of $\Sigma_5^\sharp(T)$	-1	-1	1	1	1	1
generator of $\Sigma_3^\sharp(T)$	1	1	-1	1	1	1
generator of $\Sigma_2^\sharp(T)$	1	1	1	1	1	1
$\varphi(-1,1)$	-1	1	-1	1	-1	1
$\varphi(1,-1)$	1	1	1	-1	1	-1
	δ_5	$\lvert\cdot\rvert_5$	δ_3	$\lvert\cdot\rvert_3$	δ_2	$\lvert\cdot\rvert_2$
a symmetry of $\mathbf{A}_4$	-1	-1	1	-1	1	1
a transposition $\mathbf{A}_4 \leftrightarrow \mathbf{A}_4$	-1	1	1	-1	1	1
a symmetry of $\mathbf{A}_2$	1	-1	-1	1	1	-1
a transposition $\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$	1	-1	-1	-1	1	-1

The rank of the matrix composed by the 9 rows of the table (see, Remark 4.2.3) is $6 = \dim \Gamma'_{2,0} + \dim \Gamma_{3,0} + \dim \Gamma_{5,0}$, which implies that $d^\perp$ is surjective. For the remaining two cases

$$\mathbf{D}_4 \oplus 2\mathbf{A}_4 \oplus 3\mathbf{A}_2, \quad 2\mathbf{A}_7 \oplus 2\mathbf{A}_2,$$

the computation is almost literally the same. For $2\mathbf{A}_7 \oplus 2\mathbf{A}_2$, where $\Sigma_2^\sharp(\mathbf{T}) \not\supset \Gamma_{2,2}$, we have to modify $|\cdot|_2$ by replacing χ_2 with $\chi_2(u) = u \mod 8 \in \{1,3,5,7\} = \mathbb{Z}_2^\times/(\mathbb{Z}_2^\times)^2$ and consider the full group $\Gamma_{2,0}$ instead of $\Gamma'_{2,0}$. □

4.2.3. *Remark.* Here and below, when speaking about ranks and dimensions, we regard all groups Γ_*, E, E^+, etc. as $\mathbb{F}_2$-vector spaces. In particular, when computing the rank of a matrix, we need to switch from the multiplicative notation $\{1,-1\}$ to the additive $\{0,1\}$.

4.2.4. **Corollary** (of the proof)**.** *For all sets of singularities* $\mathbf{S}$ *with* $\mu(\mathbf{S}) \le 18$*, the corresponding transcendental lattice* $\mathbf{T}$ *is unique in its genus, i.e.,* $g(\mathbf{T}) = 1$*.*

4.2.5. **Proposition.** *If* $\mathbf{S}$ *is one of*

$$\mathbf{D}_6 \oplus 2\mathbf{A}_6, \quad \mathbf{D}_5 \oplus 2\mathbf{A}_6 \oplus \mathbf{A}_1, \quad 2\mathbf{A}_7 \oplus 2\mathbf{A}_2, \quad 3\mathbf{A}_6, \quad 2\mathbf{A}_6 \oplus 2\mathbf{A}_3$$

then $\mathcal{M}_1(\mathbf{S})$ *consists of two complex conjugate components; in all other cases with* $\mu(\mathbf{S}) \le 18$*, the stratum* $\mathcal{M}_1(\mathbf{S})$ *is connected.*

Proof. By Proposition 4.2.2, $\mathcal{M}_1(\mathbf{S})$ is connected if and only if the (unique) homological type extending $\mathbf{S}$ is symmetric; otherwise $\mathcal{M}_1(\mathbf{S})$ consists of two complex conjugate components. By Theorem 2.4.4, homological type is symmetric if and only if there is an isometry $a \in O(\mathbf{T})$ with $det_+(a) = -1$ satisfying $d(a) \in d^\psi(O(\mathbf{S}))$, where d^ψ is the map induced by any anti-isometry $\psi\colon \mathcal{S} \oplus \langle \frac{1}{4} \rangle \to \mathcal{T}$. We consider separately the cases $|E(\mathbf{T})| = |E^+(\mathbf{T})|$ and $|E(\mathbf{T})| < |E^+(\mathbf{T})|$.

4.2.6. **Lemma.** *If* $|E(\mathbf{T})| = |E^+(\mathbf{T})|$*, then* $\mathcal{M}_1(\mathbf{S})$ *is connected.*

Proof. By definition, we have an exact sequence

$$0 \to \Gamma_0/\Gamma_0^{--} \cdot \tilde{\Sigma}(\mathbf{T}) \to E^+(\mathbf{T}) \to E(\mathbf{T}) \to 0. \tag{4.2}$$

Hence, $|E(\mathbf{T})| = |E^+(\mathbf{T})|$ if and only if $\tilde{\Sigma}(\mathbf{T}) \not\subset \Gamma_0^{--}$. Then, by Proposition 2.7.1, there exist a $+$-disorienting isometry of $\mathbf{T}$ inducing the identity on $\operatorname{disc}\mathbf{T}$, and Theorem 2.4.4 applies. □

4.2.7. **Lemma.** *If* $|E(\mathbf{T})| < |E^+(\mathbf{T})|$*, then* $\mathcal{M}_1(\mathbf{S})$ *is connected if and only if* $d_+^\perp\colon O(S) \to E^+(\mathbf{T})$ *is an epimorphism.*

Proof. The non-trivial element of the kernel $K := \Gamma_0/\Gamma_0^{--}\cdot\tilde{\Sigma}(\mathbf{T}) \cong \{\pm 1\}$ in (4.2) is the image under the composed map $O(\mathbf{T}) \to \mathrm{Aut}(\mathcal{T}) \to E^+(\mathbf{T})$ of any element $a \in O(\mathbf{T})$ with $\det_+(a) = -1$. Thus, $\mathcal{M}_1(\mathbf{S})$ is connected if and only if $\mathrm{Im}(d_+^\perp)\cap K \neq 0$, *i.,e.*, $\mathrm{rank}\, d_+^\perp > \mathrm{rank}\, d^\perp$ (see Remark 4.2.3). On the other hand, Proposition 4.2.2 can be recast in the form $\mathrm{rank}\, d^\perp = \dim E(\mathbf{T})$. Since $\dim E_+(\mathbf{T}) = \dim E(\mathbf{T}) + 1$, the statement follows. □

Lemma 4.2.6 implies the connectedness of $\mathcal{M}_1(\mathbf{S})$ for 2721 sets of singularities. For the remaining 151 sets of singularities, one has $|E(\mathbf{T})| < |E^+(\mathbf{T})|$. For 118 of them, one has $|E(\mathbf{T})| = 1$ and $\tilde{\Sigma}_p(\mathbf{T}) \subset \Gamma_0^{--}$ for some prime p. Since $|E(\mathbf{T})| = 1$, the map $d : O(\mathbf{T}) \to \mathrm{Aut}(\mathcal{T})$ is surjective and the isomorphism

$$\mathrm{Aut}(\mathcal{T})/O^+(\mathbf{T}) = \Gamma_0/\Gamma_0^{--}\cdot\tilde{\Sigma}(\mathbf{T}) = E^+(\mathbf{T}) = \{\pm 1\}$$

(see(4.2)) is the descent of $\det_+$, which is well-defined due Proposition 2.7.1. In most cases we can use Lemma 2.7.2 to show that there exists an element $a \in O(\mathbf{S})$ such that $\det_+(d^\psi(a)) = -1$. Namely,

- if $p = 2$, take for a a nontrivial symmetry of $\mathbf{A}_2$, $\mathbf{D}_5$, $\mathbf{E}_6$ or $\mathbf{D}_9$,
- if $p = 3$, take for a a nontrivial symmetry of $\mathbf{A}_2$, $\mathbf{D}_5$ or $\mathbf{A}_8$,
- if $p = 7$, take for a a nontrivial symmetry of $\mathbf{A}_2$.

The three sets of singularities

$$\mathbf{D}_6 \oplus 2\mathbf{A}_6, \quad \mathbf{D}_5 \oplus 2\mathbf{A}_6 \oplus \mathbf{A}_1, \quad 3\mathbf{A}_6$$

with $p = 7$ are exceptional (and are listed as such in the statement), as Lemma 2.7.2 implies $\det_+ \circ d^\psi \equiv 1$. Indeed, the image of d^ψ is generated by the reflections r_α with either

- $\alpha \in \mathcal{T}_7$, $\alpha^2 = \frac{6}{7}$ (a nontrivial symmetry of $\mathbf{A}_6$) or,
- $\alpha \in \mathcal{T}_7$, $\alpha^2 = \frac{12}{7}$ (a transposition $\mathbf{A}_6 \leftrightarrow \mathbf{A}_6$) or,
- $\alpha \in \mathcal{T}_2$ (a nontrivial symmetry of $\mathbf{D}_6$ or $\mathbf{D}_5$).

One has $\delta_7(\alpha) = |\alpha|_7 = -1$ in the first two cases and $\delta_7(\alpha) = |\alpha|_7 = 1$ in the last one.

There are 30 other sets of singularities still satisfying the condition $\Gamma_{2,2} \subset \Sigma_2^\sharp(\mathbf{T})$ and having two irregular primes, so that we can apply Lemma 2.7.3. Among them, 13 sets of singularities contain two type $\mathbf{A}_2$ points and have $(p, q) = (2, 3)$ and $|E^+(\mathbf{T})| = 4$. A nontrivial symmetry of any type $\mathbf{A}_2$ point and a transposition $\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$ map to two linearly independent elements generating $E^+(\mathbf{T})$. The remaining 17 sets of singularities are listed in Table 4, where we indicate the irregular primes (p, q), order of $E^+ := E^+(\mathbf{T})$ and a collection of isometries of $\mathbf{S}$ whose images generate $E^+(\mathbf{T})$. The set of singularities $\mathbf{S} = 2\mathbf{A}_{\mathbf{6}}\oplus 2\mathbf{A}_3$ marked as exceptional is one of the special cases listed in the statement. We have $|E^+(\mathbf{T})| = 4$ and the group $O(\mathbf{S})$ is generated by

- a nontrivial symmetry of $\mathbf{A}_3$, mapped to $(1, 1) \in E^+(\mathbf{T})$, or
- the transposition $\mathbf{A}_3 \leftrightarrow \mathbf{A}_3$, mapped to $(1, 1) \in E^+(\mathbf{T})$, or
- a nontrivial symmetry of $\mathbf{A}_6$, mapped to $(-1, 1) \in E^+(\mathbf{T})$, or
- the transposition $\mathbf{A}_6 \leftrightarrow \mathbf{A}_6$, mapped to $(-1, 1) \in E^+(\mathbf{T})$.

It follows that $d_+^\perp$ is not surjective.

Finally, what remains are the 3 sets of singularities

$$\mathbf{D}_4 \oplus 2\mathbf{A}_4 \oplus 3\mathbf{A}_2, \quad 2\mathbf{A}_7 \oplus 2\mathbf{A}_2, \quad \mathbf{A}_4 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_2$$

Table 4. Extremal singularities

Singularities	(p,q)	$\lvert E^+\rvert$	isometries of $\mathbf{S}$ generating $E^+(\mathbf{T})$
$2\mathbf{E}_6 \oplus 2\mathbf{A}_3$	$(2,3)$	4	symmetry of $\mathbf{E}_6$; $\mathbf{A}_3 \leftrightarrow \mathbf{A}_3$
$\mathbf{D}_9 \oplus \mathbf{A}_3 \oplus 3\mathbf{A}_2$	$(2,3)$	8	symmetries of $\mathbf{A}_2$, $\mathbf{A}_3$; $\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$\mathbf{D}_8 \oplus \mathbf{A}_6 \oplus 2\mathbf{A}_2$	$(2,3)$	2	$\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$\mathbf{D}_8 \oplus \mathbf{A}_3 \oplus 3\mathbf{A}_2$	$(2,3)$	2	$\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$\mathbf{D}_7 \oplus \mathbf{D}_5 \oplus 3\mathbf{A}_2$	$(2,3)$	8	symmetries of $\mathbf{A}_2$, $\mathbf{D}_5$; $\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$\mathbf{D}_7 \oplus \mathbf{D}_4 \oplus 3\mathbf{A}_2$	$(2,3)$	2	$\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$\mathbf{D}_6 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2$	$(3,5)$	2	symmetry of $\mathbf{A}_2$
$2\mathbf{D}_5 \oplus 2\mathbf{A}_4$	$(2,5)$	4	symmetry of $\mathbf{D}_5$; $\mathbf{A}_4 \leftrightarrow \mathbf{A}_4$
$\mathbf{D}_5 \oplus 2\mathbf{A}_4 \oplus 2\mathbf{A}_2 \oplus \mathbf{A}_1$	$(3,5)$	4	symmetry of $\mathbf{A}_2$; $\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$\mathbf{D}_4 \oplus 2\mathbf{A}_6 \oplus \mathbf{A}_2$	$(2,7)$	2	symmetry of $\mathbf{A}_6$
$\mathbf{A}_{11} \oplus \mathbf{A}_3 \oplus 2\mathbf{A}_2$	$(2,3)$	8	symmetries of $\mathbf{A}_2$, $\mathbf{A}_3$; $\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$\mathbf{A}_8 \oplus 2\mathbf{A}_3 \oplus 2\mathbf{A}_2$	$(2,3)$	8	symmetries of $\mathbf{A}_2$, $\mathbf{A}_3$; $\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$2\mathbf{A}_6 \oplus 2\mathbf{A}_3$	$(2,7)$	4	exceptional
$2\mathbf{A}_6 \oplus 3\mathbf{A}_2$	$(3,7)$	4	symmetries of $\mathbf{A}_2$, $\mathbf{A}_6$
$2\mathbf{A}_5 \oplus 2\mathbf{A}_4$	$(3,5)$	2	symmetry of $\mathbf{A}_4$
$3\mathbf{A}_4 \oplus 2\mathbf{A}_2 \oplus 2\mathbf{A}_1$	$(3,5)$	4	symmetry of $\mathbf{A}_2$; $\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$
$2\mathbf{A}_4 \oplus 4\mathbf{A}_2$	$(2,3)$	2	$\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$

to which Lemma 2.7.3 does not apply and we need to compute the groups $E^+(\mathbf{T})$ directly from the definition (2.7). For

$$\mathbf{S} = 2\mathbf{A}_7 \oplus 2\mathbf{A}_2,$$

which is the last exceptional case listed in statement, we have $\Sigma_2^\sharp(\mathbf{T}) \not\supset \Gamma_{2,2}$ and $|\cdot|_2$ needs to be modified by replacing χ_2 with $\chi_2(u) = u \mod 8 \in \{1,3,5,7\} = \mathbb{Z}_2^\times/(\mathbb{Z}_2^\times)^2$ and we have to consider the full group $\Gamma_{2,0}$ instead of $\Gamma'_{2,0}$. The computation can be summarized as follows:

	$\Gamma_{3,0}$		$\Gamma_{2,0}$		
generator of $\Sigma_3^\sharp(T)$	-1	-1	1	1	1
$\Sigma_2^\sharp(T) = \{1\}$	1	1	1	1	1
$\varphi(-1,-1)$	-1	-1	-1	-1	1
	δ_3	$\lvert\cdot\rvert_3$	δ_2	$\lvert\cdot\rvert_2$	
a symmetry of $\mathbf{A}_2$	-1	1	1	-1	-1
a transposition $\mathbf{A}_2 \leftrightarrow \mathbf{A}_2$	-1	-1	1	-1	-1
a symmetry of $\mathbf{A}_7$	1	1	-1	-1	1
a transposition $\mathbf{A}_7 \leftrightarrow \mathbf{A}_7$	1	-1	-1	-1	1

The rank of the matrix composed by the 7 rows of the table (see Remark 4.2.3) is $4 < \dim \Gamma_{3,0} + \dim \Gamma_{2,0}$, which implies that $d_\ddagger^\perp$ is not surjective. For the other two cases, there are three irregular primes and the computation repeats literally that at the end of the proof of Proposition 4.2.2; in both cases, the map $d_\ddagger^\perp$ turns out to be surjective. □

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